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TECHNICAL REPORT

AN OVERVIEW OF THE MATHEMATICAL THEORY OF GAMES

by

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1. Introduction. Game theory is a collection of mathematical models formulated to study decision making in situations involving conflict and cooperation. It recognizes that conflict arises naturally when various participants have different preferences, and that such problems can be studied quantitatively rather than as a mere illness or abnormality which should be cured or rectified. Game theory attempts to abstract out those elements which are common and essential to many different competitive situations and to study them by means of the scientific method. It is concerned with finding optimal solutions or stable outcomes when various decision makers have conflicting objectives in mind. The game model describes in detail the potential payoffs which one expects to occur, and it points out how one should act in order to arrive at the best possible outcome in light of the options open to one's opponents. Game theory attempts to provide a normative guide to rational behavior when in a group whose members aim for different goals. Although there is much interaction between the areas, one frequently distinguishes between the theory of games (games of skill) and other subjects such as, probability and statistics (games of chance), classical decision theory, utility theory, game experiments, and gaming or simulation. In brief a game consists of players who must choose from a list of alternatives which will then bring about expected outcomes over which the participants may have different preferences.

It is clear that the scope of game theory is rather broad and ambitious, and that the analyst must restrict himself to special cases if he hopes to obtain usable results. There are consequently many logical classifications of competitive situations. Some of the ones that have proved helpful to the

investigator are: the number of participants, the number of moves and choices, the constant-sum and general-sum cases, the cooperative and noncooperative situations, the various states of information available to the players, and the different restrictions which may be placed on side payments. In addition to these and other classifications, most game theoretic models fall into one of three general mathematical formulations or some natural extension of them. These formulations are called the extensive form, the normal form, and the characteristic function form; and they are described in some detail in Sections 3, 4, and 5, respectively. The object of this paper is to present for the nonspecialist a description of the nature of these three basic models, a glimpse at some of the known mathematical properties of these models and their solution concepts, and a selective mention of some of their areas of applications. Some technical terms will be used without definition with the hope that they are self-evident or have sufficient intuitive meaning so as to be not too misleading. Selective references are given throughout for the reader who wishes to further pursue these topics.

Although some game theory concepts have arisen over the past couple of centuries, modern game theory dates from 1944 with the publication of the monumental work Theory of Games and Economic Behavior [81] by von Neumann and Morgenstern (vN-M). This classic book describes the three forms mentioned above and it extended an earlier paper by von Neumann [78 : pp. 13-42] which appeared in 1928. Some controversy has arisen, however, as to whom should be given credit for the origins of the subject (see [41 : p. 2]). See [33] for a survey of von Neumann's contributions to the theory of games and mathematical economics. A history of game theory up to 1944 is being prepared by Morgenstern. A survey of the modern theory until 1957 is presented in

Luce and Raiffa [41], and the text by Owen [58] describes some of the more recently developed models. Several of the major mathematical papers in the field have appeared in the five issues of the Annals of Mathematics Studies [14,16,31,32,78]. The introductions to these studies also provide excellent survey material on the subject. A sign that game theory has come of age is the publication beginning this year of the International Journal of Game Theory by Physica-Verlag. The recently published Russian edition of [81] has a survey of Soviet work in the field.

2. Applications. Many of the accomplishments of game theory are of a rather general or conceptual nature. Its ideas, methodology and vocabulary have become part of the daily thinking and language of most decision makers as well as many others. Its concepts arise regularly in a broad spectrum of activity: in government conferences, in military strategy sessions, in corporation board meetings, in some of the new educational devices in the schools, in the many "adult" games which have recently flooded the market, etc. It is of course, difficult now to determine to what extent these changes were influenced by the mathematical theory of games, concurrent analytical approaches, to less quantitative methods, or to many other parallel developments. However, it is certainly clear that there has been a rather profound change in the way many people think about competitive situations, and it is likely that many of these advances owe a great deal to the formulation of the theory of games, even though many of these are now taken for granted. Many of the advances have come about due to the introduction of the scientific method to analyze problems that originate in the social area. There are certain efficiencies and insights gained by abstracting, generalizing, forgetting differences, attempting to assign quantitative measures to

concepts, and clarifying what is common to a large number of different problems. This often illuminates the nature of the problem, leads to the grasping of its essential elements, and helps to pinpoint the difficulties involved. Some of the ideas suggested by game theory are rather obvious; for example, listing all of your alternatives and likely outcomes rather than just comparing a few of the traditional or more obvious ones, or placing yourself in the position of your opponent and considering all of his possibilities rather than just his past policies. Just the formulation and understanding of the game to this extent is often a partial solution to a problem. As obvious as these ideas are there is some experimental evidence to indicate that people do not always do this. (There are still many decision makers like the old authoritarian surgeon who recognizes (or at least tells you) only three probabilities of a successful outcome depending on which operation is being contemplated: $1 - \varepsilon$, $1/2$, and ε where ε is some small nonnegative number but fixed.) Other contributions often attributed to game theory are perhaps less obvious; for example, assigning quantitative measures to the expected gains obtained from threats and bluffing, the clarification of when secrecy is essential and the potential benefits from spying, and the need for unpredicability as given by a "mixed" strategy. Even poor Charlie Brown in the comic strip of today knows the futility of the cyclic reasoning involved in trying to decide how to out-fox the hitter, that is, whether to pitch a fast ball or a slow curve. Contrast this to the decision rule used by the school boy described by Edgar Allan Poe who supposedly won all the marbles in the "even-odd" matching game [12 : p. 26]. These formerly vague concepts can be made much more precise today, and one can often actually compute the optimal frequencies with which he should bluff, mix his choices, etc. Just

the formal definition of a "nonzero-sum" game may have brought home to some people the simple fact that every loss to your enemy need not be a gain for yourself, which contrasts with the "zero-sum thinking" we observe so often concerning opposing forces in the international political scene. It is true that many of these concepts made precise in game theory were not all new at the time and appear as rather straightforward today, and consequently some tend to downplay the contributions of the originators of the subject. Nevertheless, many of these have only become "obvious" after the fact. Moreover, there is a certain trivialization of knowledge which often takes place over time. It is difficult to assess, for example, the influence of the developments of calculus in areas beyond its immediate impact on mathematics and physics. There is evidence that the game theory of the past quarter century or so has had a profound effect on the way we view problems and make decisions. However, it is not unlikely that some of the current research in the area will have similar payoff in the future.

There have been a large number of concrete applications of game theory to many different fields, including economic theory and application, operations research, business, management science, political theory, military science, education, sociology and psychology. There are hundreds of papers detailing these various uses of the game models, and any attempt to survey these or to compose a fairly complete bibliography on them would be very time-consuming and extremely lengthy. A small fraction of the areas of applicability are, however, mentioned in the succeeding sections after a general discussion of the particular game models. The Selected RAND Abstracts and the Index of Selected Publications of The RAND Corporation alone lists over 400 papers in the game theory area proper, and a good fraction of these

are relevant to applications. Presumably RAND has put out many other papers in this area which were for internal or limited distribution or were classified, and several of its other contributions made use of this theory. Many of these applications of game theory have influenced major decisions in management in several fields and at all levels. In a recent and very excellent text [82 : pp. x, xiii, 161-162] on operations research containing over 1000 pages, the author has chosen to relegate game theory to one long exercise following linear programming which outlines the zero-sum matrix games. In his preface he makes the following statement which is similar to remarks made by others before him. "In practicing operations research, we have found that game theory does not contribute any managerial insights to real competitive and cooperative decision-making behavior that are not already familiar to church-going poker players who regularly read the Wall Street Journal." I agree that it is highly desirable to not add another brief chapter to the several dozen now in existence in texts in several different subjects which cover this very specialized topic in game theory. On the other hand, I heartily disagree with the implication that new contributions of a nontransparent nature are not made regularly today by means of game theoretical analysis. In two and one-half years at The RAND Corporation where I was one of more than a dozen game theorists, I was consulted a couple of times each month about an important nonobvious real-world problem for which the game approach was the most appropriate analytical tool. In a significant percent of these cases insights or solutions could be obtained or at least approximated. The results of this type have certainly been of much interest to many managers and decision makers. Another view of how top corporate managers view many of their decisions in a game context and how they see the rules of

the game changing in today's world is given in a series of articles in Fortune magazine over the past couple of years written by John McDonald. McDonald's earlier book [50] is still an enjoyable and very readable introduction to the topic for the layman. Perhaps some of the extreme and diverse views on the applicability of game theory heard currently are brought about by the great fluctuation in the expectations of this subject over the years. Shortly after the appearance of Theory of Games and Economic Behavior the expectations for this subject skyrocketed to great heights as can be seen from some of the reviews for this book written in that period; for example, see [10]. Even before its publication, there was some consideration given to classifying this book, because of the valuable and potentially useful information it would give the enemy. These expectations plummeted back down after a few years because it became clear to some that many difficulties existed and that solution concepts for the more general games were not as sharp or precise as one may have hoped for. In short, the overly ambitious projections were followed by a deep disillusionment. There is good evidence, however, that expectations have been on the rise, though with a smaller rate of increase, for the last decade. When one hears praise or criticism of game theory he should check into which of these periods gave rise to these opinions and whether they are still valid.

There are many substantial difficulties, unsolved problems, and valid criticisms of game theory. On the other hand, one often hears many superficial or unjustified criticisms of the subject, as well as some which now are of historical interest only. There is also an impatience with the lack of the theory's use in making predictions in the real-world, but it is not clear that these are so few in comparison with some other popular analytical

approaches in light of the shortage of broadly trained experts in the area. It is quite clear that many important competitive situations are covered inadequately or not at all by current theory, and there is much need for extension of known methods as well as for the introduction of additional models. It is preferable, however, that the researchers in these new areas be thoroughly familiar with the capabilities and limitations of existing knowledge, as apparently has not always been the case. The theoretical development of games has been hindered by the inherent mathematical difficulties of the subject. The field has suffered because of the small amount of close contact between game theorists and pure mathematicians in recent years, despite the fact that many of the originating groups and a couple of the current centers are in mathematics departments. Imagine what would be the position and impact of game theory and other modern techniques in the social sciences, if they had received as much attention from high quality quantitative analysts and experimentalists over as long a period of time as has been the case in much of the physical sciences. It is unlikely that progress would be anywhere as slow as it has been, in say, differential games, if this topic got as much attention as the theory of differential equations. On the other hand, the behavioral sciences are hardly as simple in some ways as the physical sciences; for example, in the number and scope of the basic principles or laws involved. What game theory has to offer to understanding in social sciences has not been widely appreciated, and the professional game theorists are often guilty of not communicating potential applications of existing theory, in particular the more recent results. There is evidence that specialists in some fields are completely unaware of potential benefits, whereas those in other areas have some knowledge but lack the mathematical background or incentive to pursue these possibilities.

Modern game theory is a rather young science. It has been on the scene for just over a quarter of a century, and it is still in a state of active development. It has already become a standard tool for many investigators in several areas. Its broad scope is evidenced by the large number of different university departments which employ its users and teach courses on it, as well as by the variety of texts which present some of its concepts. Game theory is the only quantitative approach so far for analyzing certain types of problems. Although its proponents may have made overly ambitious projections for it initially, it is highly unlikely that we will return to a less mathematical approach to these problems in future studies. Even though most of the applications to date have been rather elementary, there is a great deal of additional theory around which should prove beneficial in many areas. In fact, theory has recently been outdistancing applications.

We close this discussion of the applicability of game theory with the opening passage from the recent book Knots (Pantheon Books) by R.D. Laing.

> They are playing a game. They
>
> are playing at not
>
> playing a game. If I show them
>
> I see they are, I
>
> shall break the rules and they
>
> will punish me.
>
> I must play their game, of not
>
> seeing I see the game.

3. The Game Tree. The first detailed description of a game in extensive form was presented by vN-M [81 : Chapter 2] in terms of set theoretic concepts. A slightly different and more geometric definition due to Kuhn [28;32 : pp. 193-216] is usually employed today, and is presented here in a somewhat intuitive manner. For a more detailed discussion and some simple examples consult [41 : Chapters 3 and 7] and the following paper by Aumann and Maschler.

A general n-person game in extensive form is a topological tree (a finite connected graph with no cycles and with no vertices of degree two) with the following specifications. There is one distinguished vertex corresponding to the starting point. Each nonterminal vertex is a choice point and is labeled with one of the n players 1, 2, ..., n; or by the "chance player" denoted by 0. Each edge "leading out" of a vertex describes a possible move by the corresponding player if this point in the game is reached. Each vertex labeled by 0 has a probability distribution over its moves. To each terminal vertex there is assigned an n-dimensional payoff vector whose components describe the outcomes to the respective players when the game ends there. The set of all vertices for a particular player i is partitioned into information sets. When it is one of his turns to make a choice, i is aware of the information set he is in, but he does not know the precise vertex within this set. For each vertex in a given information set there is the same number of potential moves, and one normally assumes that in playing a game no such vertex can come "after" another one in the same information set.

When engaged in a particular game each player is faced with the problem of how to best play the game in order to maximize his expected payoff. A

player's complete plan for playing a game is called a strategy, of which there are several different types. A pure strategy for player i is a rule for picking a particular move at each of his information sets. A mixed strategy for i is a global randomization over his pures, that is, a probability distribution over the set of all pure strategies. A behavioral strategy for i is an overall plan for local randomization; that is, a behavioral strategy consists of a class of probability distributions such that one particular distribution is assigned to the set of moves at each of his information sets. Certain combinations of "partial" pure strategies called "signaling strategies" and "associated" behavioral strategies are called "composite strategies" [32 : pp. 267-289]. The main solution concept for games in extensive form is the Nash equilibrium point which is a collection of n "optimal" strategies (one for each player) such that a unilateral change in strategy by only one player cannot possibly improve his expected payoff. The principal problem is to determine such equilibrium points for a given game.

Most of the major results about extensive games are of a theoretical nature. These theorems guarantee the existence of optimal strategies which will achieve an equilibrium. In 1912 Zermelo [88] (see vN-M [81]) demonstrated the existence of an optimal pure strategy for two-person zero-sum games with perfect information, that is, games such as checkers or chess, in which all information sets contain a single vertex. Kuhn [28;32 : pp. 193-216] extended these results about optimal pure strategies to the n-person general-sum games with perfect information. In 1928 von Neumann [78 : pp. 13-42] showed the existence of optimal mixed strategies for any two person zero sum game; this is the famous Minimax Theorem

or Fundamental Theorem of Game Theory. Nash [55, 56] extended the equilibrium concept beyond the two-person zero-sum games, and he proved that there are optimal mixed strategies in the n-person general-sum case. Kuhn also showed the existence of optimal behavioral strategies for games with "perfect recall" such as in poker. A game has perfect recall if each player is "aware" at each of his moves of precisely what moves he picked prior to it, but may not know all the choices made by the other players. Thompson [32 : pp. 267-289] proved that "composite" strategies suffice to obtain an equilibrium point for an arbitrary game, and he illustrated his theory for simplified models of bridge.

When a game arises in applications, one frequently begins his analysis by describing the game in extensive form. This gives him a rather complete picture of his problem, including the detailed structure or rules of the game, the sequence of all possible moves, the state of each player's information, and the payoffs. Except for very elementary games, however, one does not normally pursue his investigation in the extensive form. This is because the mathematical problem of actually determining optimal strategies for this form in an efficient manner has not been resolved. Instead, one reduces the game to its normal form (Section 4) by considering the full list of pure strategies for each player and the resulting payoffs when these strategies are employed. In theory it is a much simpler problem to compute optimal strategies in the normal form. For example, in the case of the two-person zero-sum games, it is equivalent to solving a pair of dual linear programs. On the other hand, in practice one usually gets an enormous increase in the number of parameters necessary to describe a mixed strategy in the resulting normal form compared to the number of variables necessary to determine a

behavioral strategy in the original game tree. For example, Kuhn [29 : pp. 101-105; 58 : p. 99] has described a trivial poker game in which the simplex of mixed strategies has dimension 8191 while the "cube" of behavioral strategies has dimension 13. This astronomical increase in the number of variables to be determined actually occurs in some important real-world problems and often forces the analyst to abandon the game theoretic approach.

Furthermore, even if the reduction to normal form results in a problem which is computationally feasible, there is still a major concern about whether this approach is philosophically sound. It is true that for games with perfect recall an optimal mixed strategy induces a behavioral strategy which will achieve the same expected payoff. It is questionable, however, whether one would or should play throughout the game according to this latter strategy. The subsequent paper by Aumann and Maschler illustrates some pitfalls in the passage from extensive to normal form and it questions some other fundamental assertions in the foundations of game theory -- at least to the extent that they are commonly understood.

In attempting to understand and simplify the theory of games in extensive form, some early workers investigated a host of questions about useless information, the relation between information partitions and the set of strategies, notions of "equivalence" of games, etc. See, for example, the paper by Dalkey [32 : pp. 217-243]. There are still many open problems in this area.

From the previous three paragraphs it is clear that there are a great number of unsolved problems in the theory of games in extensive form. Historically, research in this area has been mostly by-passed in recent years in

favor of work on the other forms. It is surely important, however, to resolve some of these outstanding and neglected problems, and the time may now be ripe for a new attack on them. For some recent work along these lines see Boudwin, Rosenthal and Wilson [7, 85].

It is easy to extend in various ways the concept of a finite game tree to the infinite case. Gale and Stewart [32 : pp. 245-266] have shown that certain infinite games with perfect information need not have a pure strategy solution if one assumes the axiom of choice. This has given rise to alternate assumptions and this topic has been applied in the basic foundations of mathematics (for example, see Mycielski [54]). Infinite games have also found application in more practical problems such as the well known bomber battleship duel (see Ferguson [19]).

4. The Normal Form. A game in normal form is simply a list of pure strategies for each one of the players along with a description of the resulting payoffs to the players for any possible strategy choices. Let the n players be denoted by $1, 2, \ldots, n$, and let player i's set of n_i pure strategies be identified with the nonempty and finite set of integers $N_i = \{1, 2, \ldots, n_i\}$. If the payoffs are in terms of real numbers, then a finite n-person game in normal form or a polymatrix game can be described technically as a payoff function F from the cartesian product $N_1 \times N_2 \times \ldots \times N_n$ into the space R^n of n-dimensional vectors with real components. If each player i picks his pure strategy j_i, then the payoff to player k is $F_k(j_1, j_2, \ldots, j_n)$, that is, the k^{th} component of $F(j_1, j_2, \ldots, j_n)$. One can include chance events in the game by adding a set N_0 for the "chance player." There is a single given probability

distribution over N_0, and one then works with expected payoffs in the natural way. One assumes that each player knows the complete lists of strategies and the resulting payoffs for all of the players, and that each one chooses a pure strategy simultaneously or in ignorance of the other's choices. The goal of each player is to maximize his expected payoff. Any two-person game can be described by a matrix with a pair of numbers for each entry, or by a pair of ordinary matrices, one for each of the players. These are referred to as the bimatrix games. A two-person "zero-sum" game requires only a single matrix in which the payoffs are to the "row" player and are the negative of those to the "column" player. This case gives rise to the well known theory of matrix games. Some simple examples of these latter cases are given in the paper by Aumann and Maschler. Elementary presentations of the theory of matrix games are given in [20, 64, 79, 84]. This topic is covered in many textbooks at various levels; for example, in [8, 15, 27, 51].

A game in normal form is a special case of a game in extensive form, and again the main solution concept, for the noncooperative games especially, is the Nash equilibrium point or some specialization of this concept. As in the general case, not all games in normal form have optimal pure strategies. Consult Dresher [17] for a discussion and for references on the frequency of pure strategy equilibria in "random" games. On the other hand, we remarked in the previous section that all games do have an equilibrium in mixed strategies, that is, probability distributions over a player's pure strategies. If each player i picks the mixed strategy $s^i = (s^i_1, s^i_2, \ldots, s^i_{n_i})$ where the s^i_j are nonnegative and sum to one, for each i, then the

expected payoff to player k is

$$E_k(s^1, s^2, \ldots, s^n) = \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} \cdots \sum_{i_n=1}^{n_n} s^1_{i_1} s^2_{i_2} \cdots s^n_{i_n} F_k(i_1, i_2, \ldots, i_n).$$

The mixed strategy n-tuple $(t^1, t^2, \ldots, t^n)$ is an equilibrium point if

$$E_k(t^1, \ldots, t^{k-1}, t^k, t^{k+1}, \ldots, t^n) \geqq E_k(t^1, \ldots, t^{k-1}, s^k, t^{k+1}, \ldots, t^n)$$

for all mixed strategies s^k for player k and for all $k \in N$. In fact the existence theorems of von Neumann and Nash are usually presented in the normal form.

Since the existence of equilibrium points is guaranteed, the main mathematical problem consists of actually determining such solutions for a particular game. As mentioned above, in the case of the two-person zero-sum games this is equivalent to solving a pair of dual linear programs. This result was observed by von Neumann and it is covered in a great number of texts and papers, for example, in [11 : Chapter 13] and [3]. Mills [53] shows that solving two-person general-sum games is equivalent to solving a special quadratic programming problem or an integer linear programming problem. For algorithms to solve this particular quadratic program, consult Lemke and Howson [34, 35] and Mangasarian and Stone [42, 43]. Several investigators are currently developing algorithms for finding solutions for the multiperson games. The subsequent paper by Howson illustrates the work in this area, as do some other recent reports by Howson [25], Rosenmuller [68], and Sobel [75]. Furthermore, Nash showed that equilibrium points are the fixed points of a

continuous mapping, and thus they exist as a consequence of certain fixed point theorems and they may be approximated by recently developed algorithms which converge to such fixed points (see Hansen [23] and unpublished work by Shapley). Dynamic approaches to equilibrium theory using systems of differential equations have been investigated by Rosen [66, 67] and Ponstein [63]. For a more complete view of work in this active area of research the reader should also consult the several additional references given in the papers mentioned above.

There are of course many extensions and generalizations of the theory of games in normal form as presented above. A player's set of pure strategies may be infinite, discrete and/or continuous; for example, the set of all positive integers, an interval of real numbers as in the two-person games "over a square," or a certain class of curves or surfaces in space as arise in the theory of differential games (see Isaacs [26]). Although optimal strategies do not always exist in such infinite games, there are a large number of very general existence theorems about the corresponding equilibria, as illustrated by the theorems of Sion [76] and of Nikaido and Iseda [57], by several papers of Ky Fan, and in the new book by T. Parthasarathy and T.E.S. Raghavan [59]. There is also a fairly extensive literature on multistage or sequentially repeated games. An introduction to this area is given in [41 : Appendix 8] and [58 · Chapter 5]. There is significant interest currently in the basic theory of repeated games of incomplete information, as introduced by Harsanyi. Mertens and Zamir [52] list some references for work on the two-person zero-sum games and Harsanyi and Selten [24] discuss the two-person bargaining case. Other fundamental changes in the basic definitions of the normal form can be made. Many of the theoretical results on

polymatrix games are valid for types of payoffs other than the set of real numbers. Walsh [83] has introduced a two-person theory in which the expected value idea is replaced by the use of a median. Marchi [44] has investigated some generalizations of the equilibrium point concept.

There have been a great number of applications of the noncooperative games in normal form. Even a very superficial survey or classification of these is impossible in only a few pages, and it will not be attempted here. Since the two-person zero-sum case is fully equivalent to the theory of dual linear programs, it is theoretically as broad in its applications. Some problems are initially formulated more naturally in one theory than in the other. The short paper by Flood presented in this issue is an interesting search game. Historically, this problem arose as an assignment problem with dimension of the order $n!$, which von Neumann in turn observed could be solved as a much smaller hide and seek game. The generalizations of the finite theory to the infinite case and to the multistage games have also found many applications to areas such as in the theory of allocation (games of partitioning), in search and duel theory (games of timing), as well as in arms control. The n-person general-sum theory has been used extensively, although the solution concept is often not as sharp in this general case. Some difficulties arise when a game has nonunique equilibrium strategies or when these give rise to different payoff values or are not "interchangeable." There are times when one may prefer or should be advised to not make use of an optimal mixed strategy. Nevertheless, equilibrium theory is an important part of modern theoretical economics as is illustrated in the forthcoming volumes by Shapley and Shubik, or by reviewing the papers in the journal <u>Econometrica</u>. For example, it is basic in bidding theory (see Griesmer,

Shubik and Levitan [21, 22]). The infinite normal form is closely related to Abraham Wald's statistical decision theory which in turn embraces much of classical statistics. The theory of the normal form also feeds back into pure mathematics where it has lead to results in fixed point theory and separation properties as illustrated by some work of Ky Fan. In short it is clear that there is a very broadly developed theory for the normal form, and that this part of game theory has had a major impact on the way the analyst views certain problems; it has become a frequently used tool to derive actual solutions in many applied areas in the past and present; and it has contributed significantly to several other theoretical subjects.

The normal form of a game is also employed in the study of two-person general-sum <u>cooperative</u> games. In such games the two players can usually both benefit by forming a coalition and taking some joint action. However, there normally remains some element of conflict which must be resolved, and several arbitration schemes and bargaining procedures have been proposed for this purpose [41 : Chapter 6; 58 : Chapter 7].

5. <u>The Characteristic Function Form</u>. A major factor in the multiperson cooperative games is that of coalition formation, and thus the maximum amount attainable by any potential coalition is of primary concern. Consequently, the starting point for most studies in the cooperative n-person games when $n > 2$ is the characteristic function formulation suggested by von Neumann [78 : pp. 13-42] in 1928 and presented in detail in vN-M [81]. An <u>n-person game</u> (N,v) <u>in characteristic function form</u> consists of a set $N = \{1, 2, \ldots, n\}$ of n <u>players</u> denoted by $1, 2, \ldots, n$, along with a <u>characteristic function</u> v which assigns the real number $v(S)$ to each nonempty subset S of players.

The value $v(S)$ measures the expected worth or power which the coalition S can achieve when its members act together. In applications the values $v(S)$ for the various coalitions S may be obvious from the precise statement of a game problem, or it may be derived from a game in normal form by taking, for example, an "optimal solution" in the two-person game between the coalition S and its complementary subset $N - S$. However, much of the detailed structure of the game in normal form may be lost in this reduction to characteristic function form, and certain assumptions about the player's utilities are necessary in order to make use of the results of the latter theory in the original normal form.

The multiperson theories frequently use the characteristic function to define a set A of n-dimensional real vectors $x = (x_1, x_2, \ldots, x_n)$ which represent the realizable distributions of wealth among the n players; player i receives the amount x_i in the distribution x. In most models, A is called the set of imputations and consists of all x satisfying

$$x_1 + x_2 + \cdot \cdot \cdot + x_n = v(N)$$

and

$$x_i \geqq v(\{i\}) \quad \text{for all} \quad i \in N.$$

The former condition is called group rationality or Pareto optimality, and the latter restriction is referred to as individual rationality. Intuitively, one can think of $v(N)$ as the largest amount available; the players will split this amount with no player taking less than his individual value $v(\{i\})$. In some models additional restrictions are placed on the function v such as "superadditivity" or "monotonicity" and sets smaller or larger than A may be

introduced. The main problem then is to determine which imputations x are more likely to occur as final outcomes of a given game. There are many approaches which one may use in attempting to resolve this problem. Hence, there have been several different models and "solution concepts" proposed to analyze multiperson cooperative games in this form. In each such model the main mathematical problems are to prove whether every game has a solution or not, to describe the nature or properties of these solution sets, and if the model proves to be applicable to find algorithms for actually determining a solution for a given game. In applications no one model proves to be "valid" or completely satisfactory for all games, nor is any model superior to the others in all instances. A few of the well-known models in the cooperative theory are discussed most briefly in the following paragraphs. More detailed descriptions of these models are presented in a nontechnical manner in [41 : Chapters 8-12] and more recently by Davis [12] and Rapoport [65], and in a more technical form by Burger [8 : Chapter 4] and Owen [58 : Chapters 8-10]. An extensive presentation by Shapley and Shubik of this area and its economic applications is in progress. Additional references are given below for the interested reader who desires to investigate some of these particular models in more detail.

The first general solution concept introduced for the cooperative games was the von Neumann-Morgenstern solution, or stable set [81]. They defined on the set A a preference relation called dominance. A coalition S is <u>effective</u> for an imputation x, or x is S-<u>effective</u>, if the total payoff in x to the players of S is not more than their value, that is, if

$$\sum_{i \in S} x_i \leq v(S) .$$

If x and y are in A, then x dominates y if there exists a nonempty subset S such that x is S effective, and each member of S prefers x to y, that is, if

$$x_i > y_i \quad \text{for all} \quad i \in S.$$

A subset K of A is a stable set if no x in K dominates any y in K, and if every imputation z not in K is dominated by some x in K. These conditions are called internal and external stability, respectively; and they can be written as

$$K \cap \text{Dom } K = \emptyset$$

and

$$K \cup \text{Dom } K = A$$

where $\emptyset$ is the empty set and where

$$\text{Dom } K = \{y \in A: x \text{ dominates } y \text{ for some } x \in K\}.$$

Early research on stable sets indicated that they might exist for all games. In fact many games have a great number of stable sets, and many such sets contain infinitely many imputations. On the other hand, developments in recent years have indicated that the set of all stable sets for a particular game may in some instances be rather restricted, and several counterexamples to long-standing conjectures concerning the nature of stable sets have been given. These developments were high-lighted by the discovery of a ten-person game for which no stable set exists [37, 38]. The author [40] has just

completed a survey of the mathematical theory of multiperson cooperative games with particular emphasis on the theory of stable sets and the core. In addition to its potential applications, there is some interest in this concept as a purely mathematical concept.

One of the simplest solution concepts is the core. The theory of the core is implicit in the theory of stable sets since the core is a subset of any stable set. The core was first studied explicitly however, in the mid 1950's by Gillies [78 : pp. 47-85] and Shapley. The <u>core</u> is defined as

$$C = \{x \in A: \sum_{i \in S} x_i \geq v(S) \quad \text{for all nonempty} \quad S \subset N\}.$$

No coalition S can protest against or block an outcome x in C on grounds that the coalition can expect more. The outcomes in the core are neutral in the sense that no group of players can force a change. If $v(N)$ is "large enough" relative to the other values $v(S)$, then the core consists of precisely those elements which are "maximal" in A with respect to the relation of dominance. The nature of the core is well known since it is a polyhedral set and it can be studied by the techniques of linear programming or by the theory of balanced sets [70]. However, the core is often not large enough to be a suitable solution concept. C may be the empty set; for example, in any "essential constant-sum" game. Even when C is nonempty it may be too small, as in "simple games with veto players" where it assigns all the wealth to the set of veto players even though they may not be dictators, that is, they may need the help of some others to achieve these payoffs. In the "treasure hunt" game, in which any two cooperating players can carry out precisely one bar of gold, C is empty if n is odd and C

is nonempty if n is even. (The necessity of a pair in order to obtain a payoff brings to mind the anecdote on page 34 in the Reader's Digest of January, 1971 in which the workers at a shoe factory near Viareggio Italy went on "strike" by producing only left-footed shoes.) In any event, the core normally does not possess the property of external stability. There may thus be outcomes outside of the core which are not dominated by any imputation in the core.

Several authors have proposed different value theories. A player's value is an a priori measure of his expected gain in a given game. Most value theories determine a unique imputation as their solution concept. This outcome is justified by arguments based upon some concept of fairness as determined by certain axioms, upon some bargaining procedure or arbitration scheme, or upon probabilistic considerations. The subsequent paper by Owen gives a new view of the original value theory by Shapley. Owen suggests ways the Shapley value can be extended or generalized, and it describes a method for approximating the Shapley value for large games.

Aumann and Maschler [78 : pp. 443-476, 470-499, 501-512] introduced several similar solution concepts called bargaining sets. These sets describe what payoffs are stable once a given coalition structure (partition of the player set N) has formed. An outcome is stable if there is no "objection" to it, or if every objection to it gives rise to a "counter objection." Roughly speaking, a player (or coalition) j can object against another i in the same coalition S when a payoff vector x is proposed, if he can find a new coalition T without i and a payoff y in which all the members of T get more. Player i can then counter object to j's objection if he can find a coalition U without j and an outcome z in which all the

members of U get their original amount in x and anyone in $T \cap U$ gets at least what he would get in the objection y. An individual outcome in a bargaining set can stand on its own, in contrast with an imputation in a vN-M stable set. In this latter theory it is the whole set which possesses a global stability or represents a standard of behavior, and not the individual imputations in a solution. Some of the bargaining sets are known to contain at least one payoff vector for each partitioning of the players into a coalition structure, and the general mathematical structure of these sets is known. In practice it is difficult to describe a bargaining set for a particular game, because of the great number of constraints required to consider all of the possible objections and counter objections. For additional material on the classical theory of bargaining sets the reader should consult [46; 47; 45: pp. 49-56, 161-169, 265-270: and 73 : pp. 39-52, 53-56], as well as the references mentioned below for the kernel and nucleolus.

Investigations into the bargaining set theory described above have lead to the defining of additional solution concepts. Davis and Maschler [13] introduced the kernel of a game which is always a nonempty subset of this bargaining set. For more details see [1, 48, 61, 76] and the reference lists in these papers. Schmeidler [69] defined the nucleolus which turns out to be a unique outcome in the kernel, and it is in the core if the latter is nonempty. The nucleolus is the imputation which successively minimizes the maximum excess (the loudest complaint) defined by

$$e(x,S) = v(S) - \sum_{i \in S} x_i$$

over all coalitions S. For those imputations at which the maximal excess is minimal, the nucleolus minimizes the next largest one, and so forth. Some properties of the nucleolus and references on it are given in [49; 5; 40 : Section 7.2].

In addition to those mentioned above, several other solution concepts have been proposed. Some of these are reasonable outcomes [41 : Sections 11.1-11.3], ψ-stability [41 : Chapter 10], and various solutions for the games in partition function form [38, 77]. The competitive equilibrium from classical economics often relates to some of the solution concepts in those games for which it is well defined. There are many natural extensions, variations and generalizations of the concepts of characteristic function, imputation space, coalition effectiveness and the various solution concepts which were discussed above for the classical model of games in characteristic function form and in which side payments are allowed. One major model is the generalization of the classical theory to the games <u>without side payments</u> in which a <u>generalized characteristic function</u> describes more general effectiveness regions than the half spaces used above and in which a more general imputation space is employed. For a survey of work on this important model see Aumann [73 : pp. 3-27] and Billera [5]. There are several extensions of the games with a finite number of players to those with infinitely many players. This allows for the use of the tools of mathematical analysis such as measure theory, and the results may hopefully approximate those for finite games with large n where the finite techniques for analysis have often been very difficult. There is considerable research in progress currently on the theory of games with a continuum of players. Aumann and Shapley [2] have an extensive work on the theory of values, and several reports have appeared

recently on the theory of the core. Some attempts to extend the various concepts in bargaining set theory to games without side payments and infinite games have so far not been as successful [6; 60; 62]. One of the mos interesting facts about some of the infinite models is that many solution concepts which are distinct for finite games converge to the same set as the number of players becomes infinite. A few attempts have been made in both the cooperative and noncooperative cases to show that the various statically defined solution concepts also have a more dynamic stability, that is, beginning with an arbitrary imputation there is a tendency to move to an outcome in a solution set (see Billera [6]). Several authors are beginning to extend the theory of differential games from two-person to multiperson situations as illustrated in Case [9] and Vincent and Leitmenn [80].

The main direction in application of the theory of multiperson cooperative games has been towards the social and behavioral sciences. The game theory model is the most natural and suitable one suggested to date for structuring theories or experiments for many of the problems that arise in these areas, at least to the extent that these problems can be quantified. Game theory has been useful in criticizing previous approaches, both constructively and destructively, and it has provided a framework for additional investigations of both a theoretical and an empirical nature. Most of the known solution concepts have found application to economics. The formulation of economic problems as games has clarified many points, and frequently the various game solutions have then added new insights. The extent of these developments will be clear when the forthcoming volumes by Shapley and Shubik are finally realized. In particular, the recent developments in the theory of games with a continuum of players may have major impact in economics --

perhaps even comparable to that of probability theory -- if certain basic problems in the foundations and of a mathematical nature can be resolved. Although games have not yet been used much for the actual prediction of real-world outcomes, and although they so far have lacked certain dynamic features, they are nevertheless proving important in economic theory. Many of the non superficial aspects of the game approach have received considerable attention in the past few years and it is very likely that we are witnessing its rebirth in mathematical economics. In political science, the multiperson theory is used to investigate voting situations and the Shapley value serves as a power index. This application is mentioned in Owen's paper. For additional discussions in this area see [4; 18; 72 : Part III]. The multiperson theory also provides models for certain ecological problems [71].

In many applications of the cooperative theory the results seem rather vague, because there are several different solution concepts and some of these are not as sharp or precise as the answers given by many other approaches; for example, the unique solution in a two-person zero-sum game. Rather than being a weakness, however, this multiplicity of solutions is for the most part inherent to the problems, and shows the richness of the theory due to the diversity of possible human behavior in a multipolar world in which concepts like "fairness" and "most preferred" mean different things to different people. It is very unlikely that all groups of people will accept any one well-defined solution concept as being universally applicable. As for the inability of the analyst to actually evaluate some of the various known solution concepts for particular games, this is due to the inherent

mathematical difficulties in the model. On the other hand, this is becoming a more active area of research and there have been significant developments along these lines in recent years. Many of these difficulties are likely to be resolved if the theory receives as much attention as have other areas of comparable importance in applied mathematics. In summary, there have been significant and promising developments in the cooperative theory and its applications and prospects for its future appear bright.

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13. ABSTRACT

A selective review for the nonspecialist of the mathematical theory of games is presented with general remarks about its applications. The extensive form, normal form, and characteristic function form are described.

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